Indian Institute of Science Education and Research Kolkata

Department of Mathematics and Statistics

M. Sc. Entrance Test 2025

Time: 3 hours | Full Marks: 80

- ALL PARTS OF A QUESTION SHOULD BE DONE TOGETHER.
- THE SOLUTION TO EACH QUESTION MUST BEGIN IN A NEW PAGE.
- CLEARLY STATE ALL RESULTS THAT YOU USE.

Problem 1 Let G be a group.

(i) Define the notion of a normal subgroup of G. [1]

(ii) Let H a subgroup of G. Consider the relation on G given by

$$\rho := \{(a, b) \in G \times G : a^{-1}b \in H\} \subseteq G \times G.$$

Show that H is a normal subgroup of G if and only if ρ is a subgroup of $G \times G$. [4]

(iii) Let H and K be two normal subgroups of a group G. If $G = HK$, show that $G/(H \cap K)$ is a group which is isomorphic to $(G/H) \times (G/K)$. [5]

Problem 2 Let S_3 be the permutation group on 3 letters.

(i) Show that the center $Z(S_3)$ of S_3 is the trivial subgroup of S_3. [3]

(ii) Let T be the set of all 2-cycles in S_3. For any automorphism $f \in \mathrm{Aut}(S_3)$, show that the map $\widetilde{f} : T \to T,\ \sigma \mapsto f(\sigma)$ is bijective. [4]

Problem 3 If $G = \langle a \rangle$ is an infinite cyclic group, show that G has exactly two distinct generators, namely a and a^{-1}. [3]

Problem 4 Let A and B be nonsingular $n \times n$ matrices over $\mathbb{C}$. Show that if $A^{-1}B^{-1}AB = cI_n$ for some $c \in \mathbb{C}$, then $c^n = 1$. [3]

Problem 5

(i) Let a, b and c be distinct real numbers. Do the vectors $(1, 1, 1), (a, b, c)$ and (a^2, b^2, c^2) form a linearly independent subset of $\mathbb{R}^3$? Justify your answer. [4]

(ii) Let $V \subset \mathbb{R}^{11}$ be a vector subspace of dimension 4 and let $\mathcal{A}$ be the family of all linear maps $T : \mathbb{R}^{11} \to \mathbb{R}^9$ each of whose nullspace contain V. Show that $\mathcal{A}$ is a real vector space and compute its dimension. [6]

Problem 6 Let A be a $n \times n$ real matrix and let A^t denote its transpose. Show that the dimension of the nullspace of A and the dimension of the nullspace of A^tA are same. [4]

Problem 7 Let A and B be $n \times n$ complex matrices with the property that $AB = 0$. Prove or disprove: *At least one eigenvector of B is also an eigenvector of A.* [4]

Problem 8

(i) Find $\lim_{n\to\infty} (1 + e^{-n})^{\log_e n}$. [4]

(ii) Let $\{x_n\}$ be a sequence of rational numbers that lies in $(-1, 1)$. Verify whether the following are true or false, justifying your answer in each case.

a. $\{x_n\}$ has a convergent subsequence with the limit in $(-1, 1)$. [2]

b. $\{x_n\}$ has at most finitely many convergent subsequences. [4]

Problem 9 Let $\{f_n\}$ be a sequence of functions with $f_n : [0, 1] \to \mathbb{R}$ for each $n \geq 1$. Suppose that f_n converge uniformly to a continuous function f on $[0, 1]$.

(i) Show that if $\{x_n\}$ is a sequence in $[0, 1]$ which converges to $x \in [0, 1]$, then $f_n(x_n)$ converges to $f(x)$ as $n \to \infty$. [3]

(ii) If for each $n \geq 1$, there exists $x_n \in [0, 1]$ such that $f_n(x_n) = 0$, show that there exists $c \in [0, 1]$ such that $f(c) = 0$. [3]

Problem 10

(i) State Rolle's Theorem. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function such that $f(1) < f(2)$ and $f(2) > f(3)$. Show that there exist $c \in (1, 3)$ such that $f'(c) = 0$. [5]

(ii) Suppose $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function such that $f''(x) \geq 0$ for all $x \in \mathbb{R}$. If f is bounded on $\mathbb{R}$, show that there exists $c \in \mathbb{R}$ such that $f(x) = c$ for all $x \in \mathbb{R}$. [5]

Problem 11

(i) Suppose $f : (0, 1) \to \mathbb{R}$ be continuously differentiable such that both $\lim_{x\to 0^+} f'(x)$ and $\lim_{x\to 1^-} f'(x)$ exist and are finite. Show that f uniformly continuous on $(0, 1)$. Is the converse true? Justify your answer. [7]

(ii) Consider the function $f : [0, 1] \to \mathbb{R}$

$$f(x) = \begin{cases} \frac{1}{n} & \text{if } x = \frac{1}{n} \text{ for } n \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\epsilon > 0$. Construct a partition P of $[0, 1]$ such that $U(f, P) - L(f, P) < \epsilon$. [6]