Indian Institute of Science Education & Research Kolkata

DMS MSc Examination

INSTRUCTIONS

You have **3 hours**. 4^{th} June 2023

- The examination is scored out of **100 points**.
- Answers without justification will receive **NO** credit.
- If you are using a result/theorem, state it clearly before using it.

Good luck!

Question 1 [5+5 points]
(i) Let f be a continuous function on $[a, b]$, $\infty < a < b < \infty$. Assume that f has double derivative on (a, b) to be denoted by f''. Suppose that the line joining $(a, f(a))$ and $(b, f(b))$ intersects the graph of f at a third point $(c, f(c))$ where $c \in (a, b)$. Prove that there exists a $t \in (a, b)$ such that $f''(t) = 0$.

(ii) For a differentiable function ϕ on $[a, b]$ suppose that there is a constant $M > 0$ such that $\phi'(t) \geq M \; \forall \; t \in [a, b]$. Prove that

$$\left| \int_a^b \sin(\phi(t)) dt \right| \leq \frac{\phi(b) - \phi(a)}{M}.$$

Question 2 [10 points]
Let $S = (0, 1) \cap \mathbb{Q}$ and $S' = (0, 1) \backslash S$ where $\mathbb{Q}$ denotes the set of rational numbers. Does there exist a continuous function $f : (0, 1) \to (0, 1)$ such that $f(S) \subset S'$ and $f(S') \subset S$?

Question 3 [6+4 points]
Consider the series

$$\sum_{n=3}^{\infty} \frac{1}{n^a (\log n)^b}.$$

(i) Prove that the above series is convergent if $a > 1$ and divergent if $a < 1$.

(ii) What will happen if $a = 1$?

Question 4 [5 points]
Let $f : \mathbb{R} \to \mathbb{R}$ be a periodic function. If $\lim_{x \to \infty} f(x)$ exists then show that f is a constant function.

Question 5 [6 points]
Let $\{a_n\}_{n \in \mathbb{N}}$ be a decreasing sequence of positive real numbers. Suppose the series $\sum_{n=1}^{\infty} a_n$ converges. Show that the sequence $\{n a_n\}_{n \in \mathbb{N}}$ converges to zero as $n \to \infty$.

Question 6 [6+3 points]
Consider the sequence of functions $f_m : [0, 1] \to \mathbb{R}$ defined by the following

$$f_m(x) := \lim_{n \to \infty} (\cos(m! \pi x))^{2n}.$$

(i) Show that the above sequence of functions converges pointwise to a function $f : [0, 1] \to \mathbb{R}$ which is not Riemann integrable on $[0, 1]$.

(ii) Is the above convergence uniform? Justify.

Question 7 [5 points]
Let $A \in M_n(\mathbb{R})$ be an invertible matrix with integer entries. Show that A^{-1} has integer entries if and only if $\det(A) = \pm 1$.

Question 8 [5 points]
Let $A, B \in M_9(\mathbb{R})$ be such that $\operatorname{rank}(A) = 3$ and $\operatorname{rank}(B) = 5$. Show that there is an $x \in \mathbb{R}^9$ such that $Ax = Bx = 0$.

Question 9 [5+5 points]
Let n be a natural number with $n \geq 2$.

(i) When n is odd, show that every invertible linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ has a one-dimensional subspace $L \subset \mathbb{R}^n$ such that $T(L) = L$.

(ii) For $n = 2$, give an example of an invertible linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ such that T does not have any one-dimensional invariant subspace. Can you do it for $n = 4$?

Question 10 [10 points]
Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and let V be a finite dimensional vector space over $\mathbb{K}$. A subset $B \subset V$ has the property that every function $f : B \to \mathbb{K}$ has a unique linear extension i.e., there exists a unique $\mathbb{K}$-linear map $\tilde{f} : V \to \mathbb{K}$ such that $\tilde{f}(x) = f(x)$ for all $x \in B$. Show that B is a basis for V.

Question 11 [5 points]
Consider the product group $\mathbb{R}^* \times \mathbb{R}$, where $\mathbb{R}^*$ is the multiplicative group of non-zero real numbers and $\mathbb{R}$ is the additive group of real numbers. Let $\mathbb{C}^*$ be the multiplicative group of non-zero complex numbers. Let $f : \mathbb{R}^* \times \mathbb{R} \to \mathbb{C}^*$ be the map defined by

$$f(r, \theta) = |r| e^{2\pi i \theta}, \quad \forall \ (r, \theta) \in \mathbb{R}^* \times \mathbb{R}.$$

Show that f is a surjective group homomorphism. Determine the kernel of f.

Question 12 [5 points]
Find all the elements of order 2 in the permutation group S_4. Determine the subgroup of S_4 generated by the elements of order 2.

Question 13 [3+2 points]
Let R be a ring with 1 and $x, y \in R$ be such that $xy = 1$ and $yx \neq 1$.
(i) Prove that $1 - yx$ is an idempotent element and $y(1 - yx)$ is a nilpotent element in R.

(ii) Show that there are infinitely many nilpotent elements in R.
[An element $a \in R$ is called *idempotent* if $a^2 = a$, and *nilpotent* if $a^n = 0$ for some $n \in \mathbb{Z}^+$]

Question 14 [3+2 points]
Let R be a commutative ring with 1 such that $x^3 = x$ for all $x \in R$.

(i) Show that any prime ideal P in R is maximal.

(ii) What is the cardinality of R/P for a prime ideal P?