Indian Institute of Science Education & Research Kolkata

DMS MSc Entrance Examination

INSTRUCTIONS

You have **3 hours**. 2nd June 2024

- The examination is scored out of **100 points**.
- Answers without justification will receive **NO** credit.
- If you are using a result/theorem, state it clearly before using it.

Good luck!

Question 1 [2+4 points]

(a) Let G be a finite group whose all proper subgroups are cyclic. Justify if G is necessarily cyclic.
(b) Is $(\mathbb{Q}, +)$ finitely generated? Justify your answer.

Question 2 [5 points]

Let G be a group. If $a \in G$ has finite order, show that $\text{ord}(a^n) = \frac{\text{ord}(a)}{\gcd(n, \text{ord}(a))}$.

Question 3 [4 points]

Show that any infinite cyclic group has exactly two generators.

Question 4 [7 points]

Let G be a finite group and let N be a normal subgroup of G of order n. If $\gcd(n, [G : N]) = 1$, show that N is the unique subgroup of G of order n.

Question 5 [8 points]

Given any two groups G and H, let $\text{Hom}(G, H)$ be the set of all group homomorphisms from G into H. If A is an abelian group, show that there is a natural bijective correspondence between the sets $\text{Hom}(S_3, A)$ and $\text{Hom}(\mathbb{Z}_2, A)$.

Question 6 [4 points]

Let V be the vector space of all functions from $\mathbb{R}$ to $\mathbb{R}$. Let $W = \{f \in V \ : \ f(\pi) = 0\}$. Does there exist a subspace $W' \subseteq V$ such that $V = W \oplus W'$? Justify your answer.

Question 7 [4+6 points]

(a) Construct a linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\text{Im}(T) = \{(x_1, x_2) \ : \ x_1 + x_2 = 0\}$, where $\text{Im}(T)$ denotes the image space of T.
(b) For each $n > 2$, is it possible to construct a linear map $T : \mathbb{R}^2 \to \mathbb{R}^n$ such that

$$\text{Im}(T) = \left\{ (x_1, x_2, \ldots, x_n) \ : \ \sum_{i=1}^{n} x_i = 0 \right\}?$$

Justify your answer.

Question 8 [5 points]

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map, whose matrix representation with respect to some basis has off-diagonal entries with the same sign. Then show that $\mathbb{R}^2$ has a basis consisting of the eigenvectors of T.

Question 9 [5+6 points]

Let A be the $n \times n$ matrix with all its diagonal entries equal to 1 and all its off-diagonal entries equal to $a \in \mathbb{R}$.

(a) Show that $\det(A) = (1 + (n-1)a)(1-a)^{n-1}$.
(b) Find all the eigenvalues and the eigenvectors of A.

Question 10 [5 points]

Let $\{a_n\}_{n \in \mathbb{N}}$ be a sequence defined by

$$a_n = \begin{cases} n, & n \in \{3, 6, 9, \ldots\} \\ n^2 \sin\left(\frac{1}{n}\right), & n \in \{1, 4, 7, \ldots\} \\ n \sin\left(\frac{1}{n}\right) + \sin(n), & n \in \{2, 5, 8, \ldots\} \end{cases}$$

Show that the sequence $\{a_n\}_{n \in \mathbb{N}}$ has a convergent subsequence.

Question 11 [6+4 points]
Let P be a non-constant polynomial. Show that
(a) $\lim_{|x|\to\infty}|P(x)| = \infty$.
(b) Hence or otherwise, show that the set $\{x \in \mathbb{R} : 1 \leq P(x) \leq 2\}$ is compact.

Question 12 [4+6 points]
(a) Show that the series $\sum_{n=1}^{\infty} \frac{1}{n}\sin\left(\frac{x}{n}\right)$ converges uniformly on $[0, 1]$.
(b) Verify the uniform convergence of the sequence of functions $f_n(x) = \frac{x^n}{1+x^n}$ on $[0, \infty)$.

Question 13 [10 points]
Show that the function $f(x) = \sin(\sqrt{x})$ is uniformly continuous on $[0, \infty)$.

Question 14 [5 points]
Let $f : [a, b] \to \mathbb{R}$ be a non-negative Riemann integrable function such that $\int_a^b f(x)dx = 0$. Show that $f = 0$ at each point of continuity of f. Is $f = 0$ at the points of discontinuity of f? Justify.

~~~~~~~~~~~~~~~~~ ∗ End of Question Paper ∗ ~~~~~~~~~~~~~~~~~
~~~~~~~~~~~~~~~~~